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2004 J. Phys. A: Math. Gen. 37 10561

(http://iopscience.iop.org/0305-4470/37/44/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 10561–10570 PII: S0305-4470(04)82556-3

On completeness of Barut–Girardello coherent states of $su_q(1, 1)$ algebra

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Received 24 June 2004
Published 20 October 2004
Online at stacks.iop.org/JPhysA/37/10561
doi:10.1088/0305-4470/37/44/007

Abstract

The completeness of the Barut–Girardello coherent states of the quantized $su_q(1, 1)$ algebra is studied. Using the inverse Mellin transform the integration measure for the resolution of the unit operator is obtained. The measure is found to be positive-definite.

PACS numbers: 02.20.Uw, 02.30.Cj, 02.30.Gp

1. Introduction

There has been much interest in applications and generalizations of the Barut–Girardello coherent states [1], which are the eigenstates of the lowering Weyl operator of the $su(1, 1)$ algebra. Many aspects of these coherent states for a class of deformed $su(1, 1)$ algebra have also been studied. In the context of q-deformed $su_q(1, 1)$ algebra [2], the Barut–Girardello coherent states have been constructed [3–5] by various methods. It has been argued [6, 7] that, in general, a coherent state needs to satisfy a minimum set of conditions: normalizability in the Hilbert space, continuity in its labelling complex variable and existence of a resolution of unity with a positive-definite weight function. The last property allows the coherent states to form a complete (actually, an overcomplete) set, and this is essential for a majority of quantum mechanical applications of these states. Recent works [6–8] in establishing the resolution of the unit operator in an ensemble of generalized coherent states have used the method of inverse Mellin transform. In a related development, in the context of the coherent states of the q-oscillator algebra, the resolution of unity in the form of an ordinary integral with a positive-definite measure function has been demonstrated [8].

The purpose of the present work is to extend these studies to the Barut–Girardello states of the $su_q(1, 1)$ algebra. In particular, we explicitly demonstrate the completeness relation satisfied by these coherent states in terms of an ordinary integral over the complex plane. Taking an inverse Mellin transform of an associated Stieltjes moment problem, we obtain the measure function as an infinite series. The series may be related to the q-deformed Bessel

0305-4470/04/4410561+10$30.00

functions [9]. This allows us to express the measure function in a closed form. The positive definiteness of the measure is established by obtaining an integral representation of the q-Bessel function of the second kind. For real arguments the integrand turns out to be positive definite.

For the purpose of setting our framework, we briefly review the Barut–Girardello coherent states of the $su_q(1,1)$ algebra in section 2. The measure function of the completeness integral is obtained in section 3, and expressed in terms of the q-Bessel functions in section 4. The positivity of the measure is also discussed therein. Some relevant remarks are made in section 5.

2. Barut–Girardello coherent states of the $su_q(1, 1)$ algebra

The undeformed classical $su(1,1)$ algebra is defined by the commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0, \tag{2.1}$$

where its generators $(K_0, K_\pm)$ obey the Hermiticity properties $K_0^\dagger = K_0, K_\pm^\dagger = K_\mp$. The Casimir element of the algebra is given by

$$C = K_0^2 - K_0 - K_+K_-. \tag{2.2}$$

For the discrete series of representations, the basis states read $\{|n,k\rangle \mid n = 0,1,2,\ldots,2k = \pm 1, \pm 2, \ldots\}$ with the irreducible representations being parametrized by a single number k: $C = k(k-1)\mathbb{I}$. An arbitrary irreducible representation is described as follows:

$$\begin{aligned} K_0|n,k\rangle &= (n+k)|n,k\rangle, \\ K_+|n,k\rangle &= \sqrt{(n+1)(n+2k)}\,|n+1,k\rangle, \\ K_-|n,k\rangle &= \sqrt{n(n+2k-1)}\,|n-1,k\rangle. \end{aligned} \tag{2.3}$$

We assume that the set of states listed above form a complete orthonormal basis.

Standard q-deformed $su_q(2)$ and the $su_q(1,1)$ algebras have been investigated extensively [2]. The commutation rules and the Hermiticity restrictions for the generators of the deformed $su_q(1,1)$ algebra read

$$[\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm, \qquad [\mathcal{K}_-, \mathcal{K}_+] = [2\mathcal{K}_0]_q, \qquad \mathcal{K}_0^\dagger = \mathcal{K}_0, \qquad \mathcal{K}_\pm^\dagger = \mathcal{K}_\mp, \tag{2.4}$$

where $[\mathcal{X}]_q = (q^{\mathcal{X}} - q^{-\mathcal{X}})/(q - q^{-1})$, and q is a real deformation parameter. Using the well-known Curtright–Zachos map [10], valid for a generic q, the representations of the q-deformed $su_q(1,1)$ algebra may be obtained via the unitary representation (2.3) of the classical $su(1,1)$ algebra. The real operator-valued mapping function $f(K_0)$, introduced by the equations

$$\mathcal{K}_0 = K_0, \qquad \mathcal{K}_+ = f(K_0)K_+, \qquad \mathcal{K}_- = K_-f(K_0) \tag{2.5}$$

explicitly reads

$$f(K_0) = \sqrt{\frac{[K_0 - k]_q[K_0 + k - 1]_q}{(K_0 - k)(K_0 + k - 1)}}. \tag{2.6}$$

The Barut–Girardello coherent state for the q-deformed $su_q(1,1)$ algebra was previously investigated [3–5] in various contexts. Being defined as

$$\mathcal{K}_-|\alpha,k\rangle = \alpha\,|\alpha,k\rangle, \qquad \alpha \in \mathsf{C}, \tag{2.7}$$

the normalized coherent state is explicitly given by

$$|\alpha,k\rangle = \mathcal{N}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q![n+2k-1]_q!}}|n,k\rangle_q, \qquad \mathcal{N}(|\alpha|^2) = |\alpha|^{1-2k} I_{2k-1}^{(q)}(2|\alpha|), \tag{2.8}$$

where $[n]_q! = \prod_{j=1}^{n} [j]_q!$, $[0]_q! = 1$. The q-deformed modified Bessel function of integer order m reads

$$I_m^{(q)}(2z) = \sum_{n=0}^{\infty} \frac{z^{m+2n}}{[n]_q![m+n]_q!}, \qquad I_{-m}^{(q)}(2z) = I_m^{(q)}(2z). \tag{2.9}$$

In the undeformed $q \to 1$ limit, definition (2.9) reduces to the standard representation of modified Bessel function. The normalized coherent states described in (2.7) are, however, not orthogonal. This may be observed from the following non-vanishing inner product:

$$\langle \beta, k|\alpha, k\rangle = \frac{I_{2k-1}^{(q)}(2\sqrt{\alpha\bar{\beta}})}{\sqrt{I_{2k-1}^{(q)}(2|\alpha|)I_{2k-1}^{(q)}(2|\beta|)}} \exp(\mathrm{i}(k-1/2)\varphi), \tag{2.10}$$

where $\varphi = \mathrm{phase}(\beta) - \mathrm{phase}(\alpha)$. In a special case the overlap function reads

$$\langle -\alpha, k|\alpha, k\rangle = \frac{J_{2k-1}^{(q)}(2|\alpha|)}{I_{2k-1}^{(q)}(2|\alpha|)}, \tag{2.11}$$

where the q-deformed Bessel function of the first kind is defined as

$$J_m^{(q)}(2z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{m+2n}}{[n]_q![m+n]_q!}. \tag{2.12}$$

We note that the reality of the overlap function (2.11) allows us, for instance, to construct a cat-type two-dimensional subspace with orthogonal bases:

$$|\pm, k\rangle = 2^{-1/2}(1 \pm \langle -\alpha, k|\alpha, k\rangle)^{-1/2}(|\alpha, k\rangle \pm |-\alpha, k\rangle). \tag{2.13}$$

Using the properties of the carrier space (2.3), the coherent states obtained in (2.8) may also be expressed in terms of operator-valued hypergeometric functions as follows:

$$|\alpha, k\rangle \sim \exp(\alpha(f(\mathcal{K}_0))^{-2}\mathcal{K}_+(\mathcal{K}_0 + k)^{-1})|0, k\rangle = {}_0F_1(\_; 2k; \alpha\,(f(\mathcal{K}_0))^{-2}\mathcal{K}_+)|0, k\rangle. \tag{2.14}$$

It is interesting to observe that the exponentiated operator $\Theta = (f(\mathcal{K}_0))^{-2}\,\mathcal{K}_+(\mathcal{K}_0 + k)^{-1}$ in (2.14) obeys a simple commutation relation with $\mathcal{K}_-$: $[\Theta, \mathcal{K}_-] = -1$. Recalling that the operator $\mathcal{K}_-$ is diagonalized by the state $|\alpha, k\rangle$, we intend to employ this mechanism for obtaining coherent states corresponding to other suitable operators such as a complex combination of $\mathcal{K}_\pm$: $u\mathcal{K}_+ + v\mathcal{K}_-$. This topic will be described elsewhere.

3. Completeness of the coherent states $|\alpha, k\rangle$

The set of Barut–Girardello coherent states $|\alpha, k\rangle$ of the $su_q(1,1)$ algebra possesses the important property of completeness, which may be expressed as a resolution of identity in the form

$$\int \mathrm{d}\mu(\alpha)\, |\alpha, k\rangle\langle\alpha, k| = \mathbb{I}, \tag{3.1}$$

where $\mathrm{d}\mu(\alpha)$ is a measure of integration in the complex α plane. In the present section, we derive a formal expression of this measure.

Using the polar decomposition $\alpha = \rho\exp(\mathrm{i}\theta)$ concurrently with our construction of the coherent state given in (2.8), we integrate on the angular variable θ to obtain

$$I_{2k-1}^{(q)}(2|\alpha|) \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} |\alpha, k\rangle\langle\alpha, k| = \sum_{n=0}^{\infty} \frac{\rho^{2n+2k-1}}{[n]_q![n+2k-1]_q!} |n, k\rangle\langle n, k|. \tag{3.2}$$

Multiplying both sides of relation (3.2) by an as yet to be determined function $F(\rho)$ and integrating over the entire complex α plane, we get

$$\int \mathrm{d}^2\alpha\, I^{(q)}_{2k-1}(2|\alpha|)F(|\alpha|)|\alpha,k\rangle\langle\alpha,k| = \sum_{n=0}^{\infty}\frac{\mathcal{I}_n}{[n]_q![n+2k-1]_q!}|n,k\rangle\langle n,k|, \tag{3.3}$$

where $\mathrm{d}^2\alpha = (2\pi)^{-1}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$, and $\mathcal{I}_n$ represents the Mellin transform of the function $F(\rho)$:

$$\mathcal{I}_n \equiv \int_0^\infty \mathrm{d}\rho\,\rho^{2(n+k)}F(\rho). \tag{3.4}$$

If we now choose the transform $\mathcal{I}_n$ in (3.3) as

$$\mathcal{I}_n = [n]_q![n+2k-1]_q! \qquad \forall n = 0,1,2,\ldots, \tag{3.5}$$

it immediately follows that by virtue of the completeness of the discrete basis states $|n,k\rangle$, the rhs in (3.3) reduces to the identity operator:

$$\int \mathrm{d}^2\alpha\, I^{(q)}_{2k-1}(2|\alpha|)F(|\alpha|)|\alpha,k\rangle\langle\alpha,k| = \mathbb{I}. \tag{3.6}$$

Comparing (3.6) with (3.1) the measure may be immediately read:

$$\mathrm{d}\mu(\alpha) = \mathrm{d}^2\alpha I^{(q)}_{2k-1}(2|\alpha|)F(|\alpha|). \tag{3.7}$$

The function $F(\rho)$ defined by the Stieltjes moment relation (3.4) may now be explicitly obtained in terms of the inverse Mellin transform as

$$F(\rho) = \frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}z\,\rho^{-(2z+\kappa)}[z-1]_q![z+\kappa-1]_q!, \qquad c>0, \tag{3.8}$$

where $z = n+1$ and $\kappa = 2k-1 \in \mathsf{Z}$. Employing the analytic continuation method, we will now determine the measure function $F(\rho)$ for non-negative integral values of κ. Later we will demonstrate in section 4 that our result may be extended to all allowed values of the representation parameter k.

The q-factorials appearing in (3.8) may be analytically continued by using the q-gamma function [9], which for the purpose of our work is defined as

$$\Gamma_q(z) = (q-q^{-1})^{1-z}\prod_{\ell=1}^{\infty}\frac{(q^{\ell}-q^{-\ell})}{(q^{z+\ell-1}-q^{-z-\ell+1})}, \qquad \Gamma_q(z+1) = [z]_q\Gamma_q(z), \qquad \Gamma_q(1) = 1. \tag{3.9}$$

The above definition yields the limiting value $\lim_{q\to 1}\Gamma_q(z)\to\Gamma(z)$, and, for a positive integer n, we obtain

$$\Gamma_q(n+1) = [n]_q!. \tag{3.10}$$

We, therefore, analytically continue (3.8) as

$$F(\rho) = \frac{1}{\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}z\,\rho^{-(2z+\kappa)}\Gamma_q(z)\Gamma_q(z+\kappa). \tag{3.11}$$

Relations (3.7) and (3.11) together provide a formal expression of the measure $\mathrm{d}\mu(\alpha)$ for resolution of unity. Evaluating the complex integral in (3.11), we will now explicitly determine the function $F(\rho)$.

To this end, we first note down the singularity structure of $\Gamma_q(z)$ in the complex plane. Using the iterated relation for an integer n,

$$\Gamma_q(z) = \prod_{\ell=1}^{n+1}[z+\ell-1]^{-1}\Gamma_q(z+n+1), \tag{3.12}$$

and expanding the rhs in powers of ε in the neighbourhood $z = -n + \varepsilon$, we obtain

$$\Gamma_q(-n+\varepsilon) = \frac{(-1)^n}{[n]_q!}\left(\frac{q-q^{-1}}{2\ln q}\right)\left[\varepsilon^{-1} + \Gamma_q'(1) + \sum_{\ell=1}^{n}\frac{q^\ell + q^{-\ell}}{q^\ell - q^{-\ell}}\ln q + \mathrm{O}(\varepsilon)\right], \tag{3.13}$$

where $\Gamma_q'(z) = \mathrm{d}\Gamma_q(z)/\mathrm{d}z$. The logarithmic derivative of the q-gamma function $\psi_q(z) = (\ln\Gamma_q(z))'$ obeys a recurrence relation

$$\psi_q(z+1) = \psi_q(z) + \frac{q^z + q^{-z}}{q^z - q^{-z}}\ln q, \tag{3.14}$$

and, consequently, it follows

$$\psi_q(n+1) = \Gamma_q'(1) + \sum_{\ell=1}^{n}\frac{q^\ell + q^{-\ell}}{q^\ell - q^{-\ell}}\ln q. \tag{3.15}$$

In the undeformed $q \to 1$ limit, we have $\psi_q(z) \to \psi(z) \equiv (\ln\Gamma(z))'$. Using the relations (3.13) and (3.15) we now obtain the singularity structure

$$\Gamma_q(-n+\varepsilon) = \frac{(-1)^n}{[n]_q!}\left(\frac{q-q^{-1}}{2\ln q}\right)[\varepsilon^{-1} + \psi_q(n+1) + \mathrm{O}(\varepsilon)]. \tag{3.16}$$

We are now in a position to explicitly evaluate the contour integral (3.11) for completely determining the integration measure (3.7). The structure of the Laurent expansion (3.16) makes it evident that the integrand in (3.11) has κ simple poles at $z = 0, -1, \ldots, -(\kappa-1)$ and infinite number of poles of order 2 at $z = -\kappa, -\kappa-1, \ldots$. The integrand vanishes exponentially as $|z| \to \infty$ on the left half-plane. Adjoining the contour in (3.11) with a semicircle $|z| = R$ on the left half-plane, and then proceeding to the limiting value of its radius, $R \to \infty$, we get the integral as a sum of the contributions arising from the above pole structure:

$$F(\rho) = 2[\mathcal{R}_I + \mathcal{R}_{II}], \qquad \mathcal{R}_I = \sum_{\text{simple}}\text{residue}, \qquad \mathcal{R}_{II} = \sum_{\text{double}}\text{residue}. \tag{3.17}$$

Using the Laurent expansion (3.16) and the structure of the integrand (3.11) the residues corresponding to both types of poles may be calculated for $\kappa > 0$ in a straightforward manner:

$$\mathcal{R}_I = \frac{q-q^{-1}}{2\ln q}\sum_{\ell=0}^{\kappa-1}(-1)^\ell\frac{[\kappa-\ell-1]_q!}{[\ell]_q!}\rho^{2\ell-\kappa}, \tag{3.18}$$

$$\mathcal{R}_{II} = (-1)^\kappa\left(\frac{q-q^{-1}}{2\ln q}\right)^2\sum_{\ell=0}^{\infty}\frac{\rho^{\kappa+2\ell}}{[\ell]_q![\kappa+\ell]_q!}(\psi_q(\ell+1) + \psi_q(\kappa+\ell+1) - 2\ln\rho). \tag{3.19}$$

For $\kappa = 0$, the integrand (3.11) does not have any simple pole; there are only poles of order 2. Consequently, for $\kappa = 0$, the rhs of (3.18) vanishes. Combining the above contributions of the residues given in (3.17)–(3.19) we obtain the promised explicit expression for the measure:

$$F(|\alpha|) = \frac{q^2-1}{q\ln q}\sum_{\ell=0}^{2k-2}(-1)^\ell\frac{[2k-\ell-2]_q!}{[\ell]_q!}|\alpha|^{2\ell-2k+1} + (-1)^{2k}\left(\frac{q^2-1}{q\ln q}\right)^2$$
$$\times\sum_{\ell=0}^{\infty}\frac{|\alpha|^{2k+2\ell-1}}{[\ell]_q![2k+\ell-1]_q!}\Big(\ln|\alpha| - \frac{1}{2}\psi_q(\ell+1) - \frac{1}{2}\psi_q(2k+\ell)\Big). \tag{3.20}$$

As noted earlier, the first term in (3.20) does not contribute for $k = 1/2$. With the results (3.7) and (3.20) in hand, the completeness relation (3.1) is fully established.

In the undeformed $q \to 1$ limit, the measure function $F(|\alpha|)$ simplifies as follows:

$$F(|\alpha|)_{q\to 1} \to 2\sum_{\ell=0}^{2k-2}(-1)^{\ell}\frac{(2k-\ell-2)!}{\ell!}|\alpha|^{2\ell-2k+1} + 4(-1)^{2k}\sum_{\ell=0}^{\infty}\frac{|\alpha|^{2k+2\ell-1}}{\ell!(2k+\ell-1)!} \times \left(\ln|\alpha| - \frac{1}{2}\psi(\ell+1) - \frac{1}{2}\psi(2k+\ell)\right). \tag{3.21}$$

The rhs in (3.21) may be expressed in terms of the classical modified Bessel function of the second kind given by

$$K_n(z) = \frac{1}{2}\sum_{\ell=0}^{n-1}(-1)^{\ell}\frac{(n-\ell-1)!}{\ell!}\left(\frac{z}{2}\right)^{2\ell-n} + (-1)^{n+1}\sum_{\ell=0}^{\infty}\frac{1}{\ell!(n+\ell)!} \times \left(\ln\frac{z}{2} - \frac{1}{2}\psi(\ell+1) - \frac{1}{2}\psi(n+\ell+1)\right)\left(\frac{z}{2}\right)^{n+2\ell}. \tag{3.22}$$

This leads to the correct classical limit [1] of the integration measure:

$$F(|\alpha|)_{q\to 1} \to 4K_{2k-1}(2|\alpha|) \Rightarrow \mathrm{d}\mu(\alpha)_{q\to 1} \to 4\mathrm{d}^2\alpha\, I_{2k-1}(2|\alpha|)K_{2k-1}(2|\alpha|), \tag{3.23}$$

where the measure function $F(|\alpha|)_{q\to 1}$ equals a multiple of the classical Bessel function of the second kind.

4. The q-Bessel functions, the measure and its positivity

In analogy with the classical result (3.23), the infinite series (3.20) obtained for the integration measure may be related to the q-Bessel functions [9]. This also allows us to extend the measure $F(|\alpha|)$ for all allowed values of k, while starting from the expression (3.20) that has been derived on the assumption that $2k$ is a positive integer.

With the above purpose in mind, we now enlist relevant results concerning the q-Bessel functions. The modified q-Bessel function of an arbitrary order ν may be obtained from (2.9) after carrying out an analytic continuation *à la* (3.10):

$$I_{\nu}^{(q)}(z) = \sum_{\ell=0}^{\infty}\frac{1}{[\ell]_q!\Gamma_q(\nu+\ell+1)}\left(\frac{z}{2}\right)^{(\nu+2\ell)}. \tag{4.1}$$

It is straightforward to show that $I_{\nu}^{(q)}(z)$ obeys the recurrence relation

$$I_{\nu-1}^{(q)}(z) - I_{\nu+1}^{(q)}(z) = \frac{2}{z}[\nu/2]_q\left(I_{\nu}^{(q)}(z\sqrt{q}) + I_{\nu}^{(q)}(z/\sqrt{q})\right). \tag{4.2}$$

Defining the q-derivative as

$$D_q f(z) = \frac{f(qz) - f(q^{-1}z)}{z(q-q^{-1})}, \tag{4.3}$$

we obtain a derivative recurrence relation

$$2D_q I_{\nu}^{(q)}(z) = q^{\pm(\nu+1)/2}I_{\nu+1}^{(q)}(zq^{\pm 1/2}) + q^{\pm(\nu-1)/2}I_{\nu-1}^{(q)}(zq^{\mp 1/2}). \tag{4.4}$$

The q-deformed Bessel function of the second kind is defined as

$$K_{\nu}^{(q)}(z) = \frac{\pi}{2\sin(\nu\pi)}\left[I_{-\nu}^{(q)}(z) - I_{\nu}^{(q)}(z)\right]. \tag{4.5}$$

The above definition ensures the property $K_{-\nu}^{(q)}(z) = K_\nu^{(q)}(z)$. The recurrence relation satisfied by $K_\nu^{(q)}(z)$ may be read from (4.2) and (4.5):

$$K_{\nu-1}^{(q)}(z) - K_{\nu+1}^{(q)}(z) = -\frac{2}{z}[\nu/2]_q\left(K_\nu^{(q)}(z\sqrt{q}) + K_\nu^{(q)}(z/\sqrt{q})\right), \tag{4.6}$$

whereas the q-derivative D_q acts, via (4.4), as follows:

$$2D_q K_\nu^{(q)}(z) = -q^{\pm(\nu+1)/2} K_{\nu+1}^{(q)}(zq^{\pm 1/2}) - q^{\pm(\nu-1)/2} K_{\nu-1}^{(q)}(zq^{\mp 1/2}). \tag{4.7}$$

We note that $I_\nu^{(q)}(z)$ and $K_\nu^{(q)}(z)$ do not satisfy identical recurrence relations and q-derivative properties. As $q \to 1$, the function $K_\nu^{(q)}(z)$ reduces to the undeformed Bessel function of the second kind $K_\nu(z)$, and, in the same limit, the measure function $F(|\alpha|)$ becomes, as given in (3.23), a multiple of $K_{2k-1}(z)$. This suggests that for the deformed case $(q \neq 1)$, there is a simple relationship with the measure function $F(|\alpha|)$ given in (3.20) and the $K_\nu^{(q)}(z)$ function for an appropriate integral value of ν. We describe next how this relation is deduced.

For a positive integral order $\nu = n$, definition (4.5) becomes indeterminate, and must be replaced as follows:

$$K_n^{(q)}(z) = \frac{(-1)^n}{2}\left(\frac{\partial I_{-\nu}^{(q)}(z)}{\partial \nu} - \frac{\partial I_\nu^{(q)}(z)}{\partial \nu}\right)_{\nu=n}. \tag{4.8}$$

An ingredient necessary for the above computation is the behaviour of the ratio $\psi_q(z)/\Gamma_q(z)$ in the neighbourhood of a pole of $\Gamma_q(z)$:

$$\frac{\psi_q(-n+\varepsilon)}{\Gamma_q(-n+\varepsilon)} = (-1)^{n+1}\frac{2\ln q}{q-q^{-1}}\,[n]_q! + \mathrm{O}(\varepsilon). \tag{4.9}$$

A somewhat lengthy calculation using the above results yields the explicit expression for the q-Bessel function of the second kind:

$$\begin{aligned} K_n^{(q)}(z) = {} & \frac{\ln q}{q-q^{-1}}\sum_{\ell=0}^{n-1}(-1)^\ell\frac{[n-\ell-1]_q!}{[\ell]_q!}\left(\frac{z}{2}\right)^{2\ell-n} + (-1)^{n+1}\sum_{\ell=0}^{\infty}\frac{1}{[\ell]_q![n+\ell]_q!} \\ & \times\left(\ln\frac{z}{2} - \frac{1}{2}\psi_q(\ell+1) - \frac{1}{2}\psi_q(n+\ell+1)\right)\left(\frac{z}{2}\right)^{n+2\ell}. \end{aligned} \tag{4.10}$$

The rhs of the above expression satisfies the recurrence relation and q-differentiation properties given in (4.6) and (4.7), respectively. This is an useful check on our derivation of the expression (4.10) of $K_n^{(q)}(z)$.

As promised previously, the measure function given in (3.20) may now be expressed as a multiple of the q-Bessel function of the second kind given in (4.10):

$$F(|\alpha|) \equiv F_{2k-1}(|\alpha|) = \left(\frac{q-q^{-1}}{\ln q}\right)^2 K_{2k-1}^{(q)}(2|\alpha|). \tag{4.11}$$

We indicated before that the above derivation has been obtained for $2k$ a positive integer. Extending (4.11) we now define the general expression of the measure function $F(|\alpha|)$ for all values of k listed in the context of (2.2):

$$F(|\alpha|) \equiv F_\nu(|\alpha|) = \left(\frac{q-q^{-1}}{\ln q}\right)^2 K_\nu^{(q)}(2|\alpha|). \tag{4.12}$$

The q-Bessel function $K_\nu^{(q)}(2|\alpha|)$ appearing in the general expression (4.12) of the measure is well defined for all allowed values of k. Using (3.7) and (4.12) the measure in the deformed case $(q \neq 1)$ is obtained as

$$\mathrm{d}\mu(\alpha) = \left(\frac{q-q^{-1}}{\ln q}\right)^2 \mathrm{d}^2\alpha\, I_{2k-1}^{(q)}(2|\alpha|)K_{2k-1}^{(q)}(2|\alpha|). \tag{4.13}$$

We now address the question of the positivity of the measure derived in (4.13). The expansion (2.9) indicates the positivity of the modified q-Bessel function $I_{2k-1}^{(q)}(2|\alpha|)$. To establish the positivity of $K_{2k-1}^{(q)}(2|\alpha|)$, we, however, need to establish its integral representation, which will be briefly sketched here. A generating function for $I_n^{(q)}(z)$ may be obtained as

$$\exp_q(zt)\exp_q(zt^{-1}) = \sum_{n=-\infty}^{\infty} I_n^{(q)}(2z)t^n, \tag{4.14}$$

where the q-deformed exponential reads

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}. \tag{4.15}$$

Using the Laurent expansion in (4.14), the function $I_n^{(q)}(z)$ for an integral order may be projected out as

$$I_n^{(q)}(2z) = \frac{1}{\pi}\int_0^{\pi} \exp_q(z\exp(\mathrm{i}\vartheta))\exp_q(z\exp(-\mathrm{i}\vartheta))\cos(n\vartheta)\,\mathrm{d}\vartheta. \tag{4.16}$$

For an arbitrary order ν, relation (4.16) may be generalized to the following contour integral in the complex t-plane:

$$I_\nu^{(q)}(2z) = \frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}} \exp_q(zt)\exp_q(zt^{-1})t^{-(\nu+1)}\,\mathrm{d}t, \tag{4.17}$$

where the integrand has a branch cut $(-\infty, 0)$ along the real axis. Choosing the contour $\mathcal{C}$ to be comprised of the segments $(\infty\ \exp(-\mathrm{i}\pi), \exp(-\mathrm{i}\pi))$, $(t = \exp(\mathrm{i}\vartheta), -\pi < \vartheta < \pi)$ and $(\exp(\mathrm{i}\pi), \infty\ \exp(\mathrm{i}\pi))$ in an anticlockwise sense, we finally obtain an integral expression for the modified q-Bessel function $I_\nu^{(q)}(z)$ for an arbitrary order ν:

$$I_\nu^{(q)}(2z) = \frac{1}{\pi}\int_0^{\pi} \exp_q(z\exp(\mathrm{i}\vartheta))\exp_q(z\exp(-\mathrm{i}\vartheta))\cos(\nu\vartheta)\,\mathrm{d}\vartheta$$
$$- \frac{\sin\nu\pi}{\pi}\int_0^{\infty} \exp_q(-z\exp t)\exp_q(-z\exp(-t))\exp(-\nu t)\,\mathrm{d}t. \tag{4.18}$$

Definition (4.5) now readily yields a suitable integral representation of the q-Bessel function of the second kind $K_\nu^{(q)}(z)$:

$$K_\nu^{(q)}(2z) = \int_0^{\infty} \exp_q(-z\exp t)\exp_q(-z\exp(-t))\cosh(\nu t)\,\mathrm{d}t. \tag{4.19}$$

To conclude about the positivity of the integrand in (4.19) for arbitrary real value of z, we need to study this property of the deformed exponential function $\exp_q(z)$. It has been demonstrated [11] that the said function may be expressed as a product of ordinary exponentials allowing us to establish the positivity of the deformed exponential (4.15) for arbitrary real arguments:

$$\exp_q(z) = \exp\left(\sum_{k=1}^{\infty} c_k(q)z^k\right), \tag{4.20}$$

where the coefficients c_k obey a linear recurrence relation

$$c_k = \frac{1}{[k]_q!} - \frac{1}{k}\sum_{\ell=1}^{k-1}\frac{\ell}{[k-\ell]_q!}c_\ell, \qquad c_1 = 1. \tag{4.21}$$

The above triangular set of linear equations may be solved up to any arbitrary order, and the first few coefficients are presented below:

$$c_2 = -\frac{(q-1)^2}{2(q^2+1)}, \qquad c_3 = \frac{(q-1)^2(q^4-q^3-q^2-q+1)}{3(q^2+1)(q^4+q^2+1)},$$
$$c_4 = -\frac{(q-1)^4(q^4-q^3-2q^2-q+1)}{4(q^2+1)(q^2-q+1)(q^4+1)}. \tag{4.22}$$

The above discussion leads us to infer that the measure function presented in (4.13) is a positive-definite quantity.

5. Remarks

To summarize, we explicitly obtained the completeness relation of the $su_q(1,1)$ Barut–Girardello coherent states in terms of an ordinary integral over the whole complex plane. The weight function of the measure in the deformed case ($q \neq 1$) is expressed in terms of a product of q-Bessel functions, and is found to be a positive-definite quantity.

Several other aspects of the problem merit attention. The map (2.8) from $\alpha \in \mathsf{C}$ to a continuous subset of unit vectors $|\alpha, k\rangle$ in the Hilbert space induces a two-dimensional surface, whose non-flat, circularly symmetric geometry is described [7] by the line element

$$\mathrm{d}\sigma_q^2 = \omega(|\alpha|^2)\,\mathrm{d}\overline{\alpha}\,\mathrm{d}\alpha = \omega(\rho^2)(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2), \tag{5.1}$$

where the polar decomposition has been used in the second equality. The metric factor may be expressed [7] as

$$\omega_q(x) = (x(\ln \mathcal{N}(x))')', \tag{5.2}$$

where $x = \rho^2$, and the normalization constant $\mathcal{N}(x)$ is given in (2.8). We have not been able to obtain a closed-form expression for the metric factor in the case of the state vectors (2.8). In the classical $q \to 1$ limit, however, the metric factor may be readily expressed in terms of the modified Bessel function:

$$\omega_{q\to 1}(x) = \frac{\mathbf{I}_{2k}(x)}{\mathbf{I}_{2k-1}(x)} + x\left(\frac{\mathbf{I}_{2k+1}(x)}{\mathbf{I}_{2k-1}(x)} - \left(\frac{\mathbf{I}_{2k}(x)}{\mathbf{I}_{2k-1}(x)}\right)^2\right), \tag{5.3}$$

where we have defined $\mathbf{I}_m(x) \equiv x^{-m/2}\, I_m(2\sqrt{x})$, and the classical modified Bessel function $I_m(2\sqrt{x})$ may be read from (2.9). Evidently, from (2.8), it follows $\mathbf{I}_{2k-1}(x) = \mathcal{N}(x)$. For the q-coherent state vectors (2.8) the metric factor may be evaluated in the limit $x \ll 1$:

$$\omega_q(x) = [2k]_q^{-1}(1 + \mathsf{c}_1 x + \mathsf{c}_2 x^2 + \cdots),$$
$$\mathsf{c}_1 = 2\left(\frac{2}{[2]_q![2k+1]_q} - \frac{1}{[2k]_q}\right),$$
$$\mathsf{c}_2 = 3\left(\frac{3}{[3]_q![2k+2]_q[2k+1]_q} - \frac{3}{[2]_q![2k+1]_q[2k]_q} + \frac{1}{[2k]_q^2}\right). \tag{5.4}$$

The evaluation of the metric factor in the non-flat cases discussed above may be useful in obtaining a path integral expression of the kernels involving these coherent states. Investigations on the metric and the curvature scalar on the two-dimensional space spanned by q-coherent state vectors were performed [12] earlier.

Lastly, we observe that the state (2.8) may be viewed as a nonlinear coherent state of a composite operator in the classical $su(1,1)$ algebra. In these applications, various expectation values of polynomials of Hermitian operators of the $su(1,1)$ algebra are needed. These expectation values may be expressed in terms of the normalization factor $\mathcal{N}(x)$ given in (2.8):

$$\langle\alpha, k|K_+^r K_-^r|\alpha, k\rangle = \frac{x^r}{\mathcal{N}(x)}(\partial_x(x\partial_x + 2k - 1))^r \mathcal{N}(x). \tag{5.5}$$

The above result may be generalized for non-Hermitian operators ($r \neq s$) as follows:

$$\langle\alpha, k|K_+^r K_0^p K_-^s|\alpha, k\rangle = \frac{\overline{\alpha}^r \alpha^s}{\mathcal{N}(|\alpha|^2)} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{[n]_q![n+2k-1]_q!}\,(n+k)^p \sqrt{P_{n,r}P_{n,s}}, \tag{5.6}$$

where

$$P_{n,r} = \prod_{\ell=1}^{r} \frac{(n+\ell)(n+\ell+2k-1)}{[n+\ell]_q[n+\ell+2k-1]_q}. \tag{5.7}$$

Acknowledgments

We thank R Jagannathan for fruitful discussions. Part of the work was done when one of us (RC) visited Institute of Mathematical Sciences, Chennai, India. RC was partially supported by the grant DAE/2001/37/12/BRNS, Government of India.

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